Boundary Points in Real Topological Semigroup Acts

By

CLARK R. McGRANERY

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
1972

ACKNOWLEDGEMENTS

The conjecture that the central result of this paper is true, namely that an element maximal but not minimal in the state-space of a real normal clan act must belong to the boundary, is due to Professor Dave Stadtlander. We discussed this for several months before I came to believe it and was eventually able to prove it. His profound knowledge of topological semigroup acts, his willingness to accept my slow beginnings and to build on them, and his interest in the progress of my research have been essential to what I have done. Professor A. D. Wallace has chaired the committee supervising me, has shared his great experience, wisdom, and valuable time each week, and provided at one time a forum, an incentive, and a source of instruction in the twice-weekly seminar he conducts. Professor Kermit Sigmon has at all times been accessible, and has made several helpful comments and suggestions about my proofs. These men have together provided that most elusive object, an atmosphere where inquiry can grow. My thanks extend also to Professors E. C. Nummela, Thomas Bowman, and Larry Hench whose participation in the committee supervising my research has been at all times most helpful.

TABLE OF CONTENTS

LIST OF FIGURES

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida in
Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

BOUNDARY POINTS IN REAL TOPOLOGICAL SEMIGROUP ACTS

By

Clark R. McGranery

June, 1972

Chairman: A. D. Wallace
Major Department: Mathematics

This paper is an investigation into the boundary points of real topological semigroup acts (S, X). In this pair, S is the topological semigroup; X is the topological space on which S acts, in the sense that there is a continuous function, F, from S x X onto X such that F(s, (t, x)) = F(s·t, x) for x, t in S and x in X. An act is said to be a real act if X is in R^n for some positive integer n; a normal act if S is normal, i.e., sS = Ss for any s in S; a clan act if S is a clan. Given an act, a quasi-order is defined on the state-space in a natural way. The main result is that an element maximal but not minimal in the state-space of a normal real clan act lies in the boundary. In order to prove this, some theorems concerning linking invariants in A. S. cohomology are first developed. There follow some less complete results if the semigroup is no longer assumed to have an identity. Various theorems about the internal structure of a state-space in a normal act are proven.

INTRODUCTION

This chapter is intended as a background to the history of the main problem being considered, a summary of some key results obtained in these investigations and an indication of some limits on extending these results. I suppose that the first results were achieved by A. D. Wallace in the early fifties when he showed that, given a clan S (i.e., a compact connected semigroup S with identity), with a proper minimal ideal such that S is contained in R^n for some positive integer n, the group of units must lie in the boundary. He proceeded to prove this in (9) in a special case, and then in (10) in the general case. This was later extended (3, Chap. B, Sect. 6, Prop. 15) to the proposition that each element of the group of units is marginal. Let me present here what is meant by a point x in a topological space X being marginal, an idea due to Wallace: namely that, given any open neighborhood U of x, there exists an open neighborhood V of x contained in U such that $H^*(X, X-V) = 0$. H^* will designate in this paper the Alexander-Spanier cohomology groups with co-efficients in an abelian group G, usually understood. Singular cohomology groups will be denoted by h^*; singular homology

groups by H_*. Reduced groups will be denoted in each case by placing a bar above the appropriate groups, as in $\bar{H}^*$

I think it appropriate to indicate the connection between marginality and being in the boundary. Suppose x is an element of a compact subset X of R^n. Then x belonging to the boundary is equivalent to the cohomological statement: given an open neighborhood U of x, there exists an open neighborhood V of x contained in U such that $H^n(X, X-V) = 0$. Thus, we see that a point's being marginal is the more general of these two concepts, and, in fact, if an element x of a compact space X is marginal, then whenever X is imbedded in R^n, x must belong to the boundary.

Now, let us consider an analogous question for semigroup acts. Our point of origin is a topological semigroup S acting on X contained in R^n. By this, of course, we mean that we have a continuous function sending S x X onto X in such a way that $s(tx) = (st)x$, where s, t belong to S and x belongs to X, and the function is denoted simply by juxtaposition, as is customary. In practice, this should not lead to ambiguity, even though multiplication in the semigroup is denoted in the same way. In order to consider an analogous question, we employ the well-known concept of a maximal element in a semigroup act. This can be put in two ways. First, a definition: given a semigroup act (S, X), an element x of X is maximal iff $x = sy$ where $s \in S$ and $y \in X$ implies that there exists $t \in S$ such that $tx = y$.

The second way of thinking of maximal elements is by means of the natural quasi-order which the act induces on X; we define $x_1 \leq x_2$ iff either there exists s in S such that $x_1 = sx_2$ or $x_1 = x_2$. One can check, as is well known, that this produces a quasi-order. Now, the maximal elements are simply those elements of X which are maximal in the quasi-order ($\leq$). The intersection of this quasi-order and its inverse is an equivalence relation $L = (\leq) (\leq)^{-1}$ on X, and the set of elements L-equivalent to x is denoted by L(x); this corresponds to Stadtlander's delta-class notation (8). If one thinks of a clan acting on itself via the semigroup multiplication, then clearly the maximal elements are precisely the group of units.

With this in mind, then, it is natural to inquire if, given a clan S acting on X in R^n and x maximal but not minimal in X,

1) Must x be marginal in X?

2) Must x be in the boundary of X?

To the first question, Stadtlander showed that the answer is no by constructing a counterexample, as seen on the next page. It should be noted that Madison and Lawson constructed the same example independently.

The state-space X consists of a sequence of concentric cones approaching a base cone as limit, together with a wind on each cone other than the base cone, where the initial points x_i of the winds form a sequence converging to x in

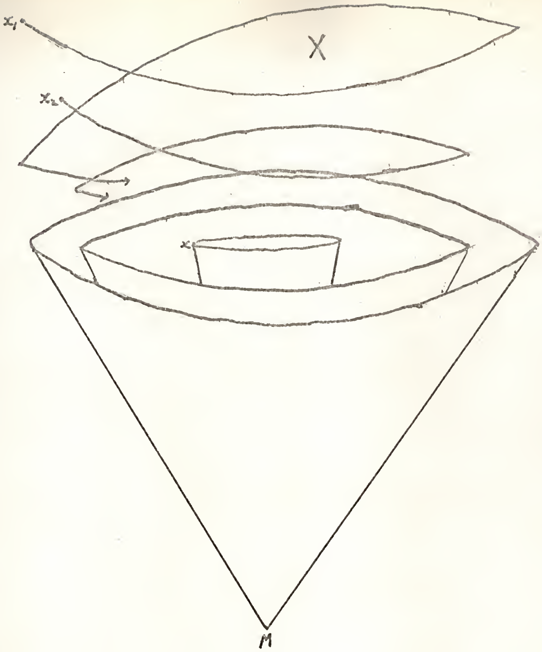

FIGURE I

the base cone. As a semigroup acting on this, we take a cone having a wind with an initial point. This is a clan; the initial point of the wind is the identity and also the entire group of units. We let this semigroup act on each cone with wind just as it would on itself, and on the base cone just as it would on its own cone. The minimal ideal of S is the vertex of the cone, and S is in fact irreducible between its minimal ideal and its group of units. Now, the set of minimal elements in X consists just of the point M, the common vertex of our cones. It is easily seen that x is maximal but not minimal, and is not marginal in X, because if we excise a small enough neighborhood of x, what remains is no longer connected and thus its cohomology differs in dimension o from that of X.

Now, as to the second question, Stadtlander conjectured that, if S is a normal clan, a point maximal but not minimal in X must be in the boundary. Such is, in fact, the case, and the proofs appear in the first three chapters of this paper.

There are additional results along these lines in what follows. For instance, one can omit the hypothesis that the semigroup S has an identity, and the result is still true under some conditions. The theorems developed to demonstrate that an element maximal but not minimal must lie in the boundary of the state-space also tell us some things about what the state-space must look like near

that point. Pretty much the same results are then obtained if R^n is replaced by S^n, the n-dimensional unit sphere. In Chapters VII - XIV, various results concerning semigroup acts are proved. It is shown that, in a normal compact act, sets of the form sX for s in S satisfy the finite intersection property and intersect in K(S)X, when K(S) is the minimal ideal of S. A result of Wallace is employed to show that, in a clan act on a state-space X with $cd(X) = n$ and $H^n(X) \neq 0$, $sX \cap tX \neq \emptyset$, for any pair of elements s, t in S. The codimension of a space X, cd(X) is the least nonnegative integer n such that, for every closed $A \subseteq X$, inclusion induces an epimorphism, $H^n(X) \rightarrow H^n(A)$.

Again, in a normal clan act, the function φ , defined by Stadtlander (8), is proven to be lower semi-continuous, but not in general continuous, and L-classes are shown to satisfy several convergency properties of a more restricted sort than, given a net x_a converging to x, $L(x_a)$ converges to L(x).

A counterexample to this last statement is provided. Some theorems of Jane Day are considered, which provide more information about normal clan acts on a continuum X having $cd(X) = n$ and $H^n(X) \neq 0$. We find among other things that, for the minimal ideal K(S) to act transitively on K(S)X, it is necessary and sufficient that some orbit have non-trivial n-th cohomology groups. We consider one result relating the codimension of an L-class to the codimension of the semigroup

S, and another showing the L-class of a maximal element to be either nowhere dense or else the entire state-space X. Some theorems of Hosszu (4) are applied to untopologized semigroup acts. The analogue of the Wedge Theorem for acts is formulated, and its use in classifying state-spaces of normal clan acts in R^2 is indicated.

CHAPTER I

An Excursion in Algebraic Topology

Throughout this paper, H^* will signify the A. S. cohomology groups, H_c^* the A. S. cohomology groups with compact support, h^* the singular cohomology groups, H_* the singular homology groups. F will, in general, represent a field, and G an abelian group in connection with coefficients for these homology and cohomology theories. In this chapter, x is an element of A, a compact subset of R^n for some positive integer n, and V is a neighborhood of x in R^n. A bar above a cohomology or homology theory will mean that it is reduced.

1.1 Theorem. Let $i\colon V-A \to R^n - A$. If

$$i_*\colon \bar{H}_j(V-A;\ G) \to \bar{H}_j(R^n - A;\ G)$$

is not trivial, then there exists a compact subset B_1 in V-A such that

$$i_1\colon B_1 \to R^n - A \qquad \text{and}$$

$$i_{1*}\colon \bar{H}_j(B_1;G) \to \bar{H}_j(R^n - A;\ G)$$

is not trivial.

Proof. Let $\overline{Z}_j(V-A;\ G)$ be the reduced group of cycles. If z is in $\overline{Z}_j(V-A;\ G)$, then $\overline{z}$, the coset of z mod $\overline{B}_j(V-A;\ G)$, is in $\overline{H}_j(V-A;\ G)$.

Since i_* is non-trivial, we may choose z_1 in $\overline{Z}_j(V-a;\ G)$ such that

$$i_*(\overline{z}_1) \neq 0.$$

Let B_1 be the support of z_1. Then B_1 is compact. Let i_2: $B_1 \longrightarrow V-A$.

Clearly there exists z_2 in $\overline{Z}_j(B_1;\ G)$ such that

$$i_{2*}(\overline{z}_2) = \overline{z}_1.$$

Then, $i_{1*}(\overline{z}_2) = i_* i_{2*}(\overline{z}_2) \neq 0$ since

$$i_1 = ii_2.$$

1.2 Theorem. If i_*: $\overline{H}_j(V-A;\ F) \longrightarrow \overline{H}_j(R^n - A;\ F)$ is not trivial, then there exists a compact polyhedron B in V-A such that

$$i_1^*:\ \overline{H}^j(R_n - A;\ F) \longrightarrow \overline{H}^j(B;\ F)$$

is not trivial. The converse is also true.

Proof. Let B be a compact polyhedron such that $B_1 \subseteq B \subseteq V-A$. (7, Ch. 6, Sec. 2, Proof of Cor. 15)

Clearly, we have $i_{1*}: \bar{H}_j(B; F) \to \bar{H}_j(R^n - A; F)$ is not trivial by 1.1 and the fact that $B_1 \subseteq B \subseteq R^n - A$.

Thus, there exists $\bar{z}$ in $\bar{H}_j(B; F)$ such that

$$i_{1*}(\bar{z}),\ \text{in } \bar{H}_j(R^n - A; F),\ \neq 0.$$

And $\mathrm{Hom}(i_*)$: $\mathrm{Hom}(\bar{H}_j(R^n - A; F), F) \longrightarrow \mathrm{Hom}(\bar{H}_j(B; F), F)$ is not trivial since the existence of an element of $\mathrm{Hom}(\bar{H}_j(R^n - A; F), F)$ which is non-zero on $i_{1*}(\bar{z})$ follows from the fact that a module over a field is free.

Furthermore, since the coefficients are a field, the usual pairing (2, Part III, Sect. 23, Cor. 23.13)

$H_j(X; F) \times h^j(X; F) \longrightarrow F$ gives an isomorphism: $\mathrm{Hom}(H_j(X; F), F) \cong h^j(X; F)$, natural in the sense that if

$$i_o: Y \longrightarrow X$$

is the inclusion, then we have the following.

<u>Lemma I</u>: The diagram

$$\begin{tikzcd}
h^j(X; F) \arrow[r, "\cong"] \arrow[d, "i_o^*"] & \mathrm{Hom}(H_j(X; F), F) \arrow[d, "\mathrm{Hom}(i_{o*})"] \\
h^j(Y; F) \arrow[r, "\cong"] & \mathrm{Hom}(H_j(Y; F), F)
\end{tikzcd}$$

commutes. (2, <u>Ibid.</u>)

Now, the same is true if we replace the singular homology and cohomology theories by the reduced singular homology and cohomology theories respectively.

In order to prove this, recall that $\overline{H}_o(X; F)$ is defined as the kernel of $C_*: H_o(X; F) \longrightarrow H_o(P; F)$, where C is the constant map from X to the space consisting of a single point P in X. On the other hand, $\overline{h}^o(X; F)$ is the cokernel of $C^*: h^o(P; F) \longrightarrow h^o(X; F)$.

If w is in $C^*(h^o(P; F))$, w takes the same value f_1 at each point of X. If v is in $H_o(X; F)$, v takes value g_i at x_i

$$1 \leq i \leq n \qquad \text{and}$$

$$g_1 + g_2 + \ldots + g_n = 0.$$

Thus, $v \times w = f_1 g_1 + f_1 g_2 + \ldots + f_1 g_n$

$$= f_1(g_1 + g_2 + \ldots + g_n) = f_1 0 = 0.$$

So, $\overline{H}_o(X; F) \times C^*(\overline{h}(P; F)) = 0$.

Therefore, the pairing

$$\overline{H}_o(X; F) \times \overline{h}^o(X; F) \longrightarrow F$$

is well defined and determines a homomorphism

$$f: \overline{h}^o(X; F) \longrightarrow \mathrm{Hom}(\overline{H}_o(X; F), F).$$

Suppose w is in $\overline{h}^o(X; F)$ and

$$\overline{H}_o(X; F) \times w = 0.$$

Let g be the coset projection from $h^0(X; F)$ to $\bar{h}^0(X; F)$, which is onto.

Take w_1 in $h^0(X; F)$ such that

$$g(w_1) = w.$$

Then $\bar{H}_0(X; F) \times w_1 = 0$. A simple argument shows that w_1 must take the same value at each point in X. Hence, w_1 is in $C^*(h^0(P; F))$ and $w = 0$. This proves that f is one-to-one.

Let h be in $\mathrm{Hom}(\bar{H}_0(X; F), F)$.

Let l: $\bar{H}_0(X; F) \longrightarrow H_0(X; F)$ be the inclusion.

There exists h_1 in $\mathrm{Hom}(H_0(X; F), F)$ such that

$$\mathrm{Hom}(l)(h_1) = h$$

since $H_0(X; F)$ is free over F.

Let h_1 correspond to w_1 in $h^0(X; F)$ under the natural isomorphism $h^0(X; F) \cong \mathrm{Hom}(H_0(X; F), F)$.

Take $w = g(w_1)$.

Then $f(w) = h$.

This proves that f is onto.

Since it is clear that naturality is preserved, the pairing

$$\bar{H}_0(X; F) \times \bar{h}^0(X; F) \longrightarrow F$$

is defined and gives a natural isomorphism

$f: \; \bar{h}^0(X; F) \cong \mathrm{Hom}(\bar{H}_0(X; F), F)$. We have proven

Lemma II: The pairing

$$\bar{H}_j(X; F) \times \bar{h}^j(X; F) \to F$$

is defined and gives a natural isomorphism

$$\bar{h}^j(X; F) \cong \mathrm{Hom}(\bar{H}_j(X; F), F).$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \bar{h}^j(R^n - A; F) & \cong & \mathrm{Hom}(\bar{H}_j(R^n - A; F), F) \\ \downarrow i_1^* & & \downarrow \mathrm{Hom}(i_{1*}) \\ \bar{h}^j(B; F) & \cong & \mathrm{Hom}(\bar{H}_j(B; F), F). \end{array}$$

Thus, we see that $i_1^*: \; \bar{h}^j(R^n - A, F) \longrightarrow \bar{h}^j(B; F)$ is not trivial since $\mathrm{Hom}(i_{1*})$ is not trivial.

Since the reduced singular cohomology theory coincides with the reduced A. S. cohomology theory on homologically locally connected spaces,

$$i_1^*: \; \bar{H}^j(R^n - A; F) \longrightarrow \bar{H}^j(B; F) \qquad \text{is not trivial.}$$

<u>1.3 Theorem</u>. If there exists an integer K and a basis of open n-balls $(V_p)_{p \text{ in } P}$ for the neighborhood system at x and some index set P such that

$$H^K(A; F) \neq 0 \qquad \text{and}$$

$$H^K(A - V_p; F) = 0 \qquad \text{for all p in P,}$$

then, given any neighborhood W of x, there exists a compact polyhedron B in W-A such that

$$i_1^*: \bar{H}^{n-K-1}(R^n - A; F) \longrightarrow \bar{H}^{n-K-1}(B; F)$$

is not trivial.

Proof. We shall consider a portion of the exact reduced singular homology sequence after selecting V_p contained in W.

$$\bar{H}_{n-K-1}(V_p - A; F) \longrightarrow \bar{H}_{n-K-1}(R^n - A; F) \longrightarrow \bar{H}_{n-K-1}(R^n-A, V_p-A; F).$$

Now

$$H^K(A; F) \cong H_{n-K}(R^n, R^n - A; F) \cong \bar{H}_{n-K-1}(R^n - A; F)$$

by (7, Ch. 6, Sec. 9, Thm. 10).

Thus, $\bar{H}_{n-K-1}(R^n - A; F) \neq 0$.

By the same theorem in Spanier, we see that

$$H_{n-K-1}(R^n - A; V_p - A; F) \cong H_c^{K+1}((R^n - A) - (V_p - A); F).$$

I shall prove that

$$H_c^{K+1}((R^n - A) - (V_p - A; F) = 0.$$

First, note that

$$(R^n - A) - (V_p - A) = (R^n - V_p) - (A - V_p).$$

Thus,

$$H_c^{K+1}((R^n - A) - (V_p - A); F) = H_c^{K+1}((R^n - V_p) - (A - V_p); F).$$

Now, by (5, p. 70),

$$H_c^i((R^n - V_p); Z) = 0 \qquad \text{for all } i \geq n.$$

By the Universal Coefficient Theorem for A. S. cohomology with compact supports, (7, Ch. 6, Sec. 8, Cor. 12), we see that

$$H_c^i(R^n - V_p; F) = 0 \qquad \text{for all } i \geq n.$$

By (5, p. 55)

$$H_c^i(R^n; Z) = \begin{cases} 0 \text{ if } i \neq n \\ Z \text{ if } i = n, \end{cases}$$

and

$$H_c^i(V_p; Z) = \begin{cases} 0 \text{ if } i \neq n \\ Z \text{ if } i = n. \end{cases}$$

Applying the Universal Coefficient Theorem above, we see that

$$H_c^i(R^n; F) = \begin{cases} 0 \text{ if } i \neq n \\ F \text{ if } i = n, \end{cases}$$

and

$$H_c^i(V_p; F) = \begin{cases} 0 \text{ if } i \neq n \\ F \text{ if } i = n. \end{cases}$$

Now, a simple application of the exact sequence for A. S. cohomology with compact supports

$$H_c^i(V_p; F) \longrightarrow H_c^i(R^n; F) \longrightarrow H_c^i(R^n - V_p; F) \longrightarrow H_c^{i+1}(V_p; F)$$

shows that

$$H_c^i(R^n - V_p; F) = 0 \qquad \text{if } i \leq n-2.$$

In order to prove that

$$H_c^{n-1}(R^n - V_p; F) = 0$$

consider that in

$$H_c^{n-1}(R^n; F) \longrightarrow H_c^{n-1}(R^n - V_p; F) \longrightarrow H_c^n(V_p; F) \longrightarrow H_c^n(R^n; F)$$

we have

$$0 \rightarrow H_c^{n-1}(R^n - V_p; F) \rightarrow F \rightarrow F \rightarrow 0.$$

Thus,

$$H_c^{n-1}(R^n - V_p; F) = 0.$$

And

$$H_c^i(R^n - V_p; F) = 0 \qquad \text{for all integers } i.$$

If we consider the exact sequence

$$H_c^K(R^n - V_p; F) \longrightarrow H_c^K(A - V_p; F) \longrightarrow H_c^{K+1}((R^n - V_p) - (A - V_p; F) \longrightarrow H_c^{K+1}(R^n - V_p; F),$$

we see that the extreme terms are zero and that, as a result,

$$H_c^{K+1}((R^n - V_p) - (A - V_p); F) \cong H_c^K(A - V_p; F),$$

which is zero, by hypothesis.

Returning to our original exact sequence

$$\begin{array}{ccc} \bar{H}_{n-K-1}(V_p - A; F) & \longrightarrow & \bar{H}_{n-K-1}(R^n - A; F) \\ & & \Big\downarrow i_* \\ & & \bar{H}_{n-K-1}(R^n - A, V_p - A; F), \end{array}$$

we have shown that the middle term is not zero and the third term is zero; whence we may conclude that i_* is not trivial. By applying 1.2, we obtain a compact polyhedron B in $V_p - A$ such that i_1^*: $\bar{H}^{n-K-1}(R^n - A; F) \longrightarrow \bar{H}^{n-K-1}(B; F)$ is not trivial.

CHAPTER II

The Main Theorem

If S is a clan acting on $X \subseteq R^n$, we call the pair (S, X) a real clan act; here and everywhere else in this paper, the action function

$$S \times X \longrightarrow X$$

is assumed to be onto; then the identity of S acts as an identity on X. In this chapter, $A = L(x)$.

2.1 Theorem. Let (S, X) be a real clan act, x be maximal but not minimal in the natural quasi-order; if there exists an integer K, an index set P, and a basis of open n-balls $(V_p)_{p \text{ in } P}$ for the neighborhood system at x such that

$$H^K(A; F) \neq 0 \qquad \text{and}$$

$$H^K(A - V_p; F) = 0 \qquad \text{for all } p \text{ in } P,$$

then x is in the boundary of X.

Proof. Let us suppose the contrary, namely that x is in the interior of X.

Then there exists a neighborhood W of x in R^n such that $W \subseteq X$. By 1.3, there exists V_p in W and a compact

polyhedron B in V_p - A such that

$$i_1^*: \bar{H}^{n-K-1}(R^n - A; F) \longrightarrow \bar{H}^{n-K-1}(B; F)$$

is not trivial, where i_1: $B \longrightarrow R^n - A$. Since x is not minimal, there exists y in X-A and s in S such that sx = y.

Let d be the usual distance function in R^n.

Then $d(y, A) = d_1 \geqslant 0$ since A is compact and hence closed.

Let $B(y; d_1/2)$ be the closed n-ball of radius $d_1/2$ about y.

V_p could clearly have been chosen so that $s \cdot V_p$ is in $B(y; d_1/2)$ by the continuity of the act, and we shall assume that this has been done.

Consider the following maps where

i_4: $SB \longrightarrow R^n - A$ is the inclusion, well defined since $A = L(x)$ for x maximal, and B is in $R^n - A$.

i_3: $sB \longrightarrow R^n - A$ is the inclusion.

i_2: $sB \longrightarrow SB$ is the inclusion.

i : $B \longrightarrow SB$ is the inclusion.

s : $B \longrightarrow sB$ is the action of s in S on B.

By considering a restriction of the act to $S \times B \longrightarrow SB$, one sees that i: $B \longrightarrow SB$ is induced by applying the identity 1 in S.

The Homotopy Lemma yields that

$$i^* = (i_2 s)^* = s^* i_2^*.$$

Let

$$j_2 \colon \; B(y_1 ; d_1/2) \longrightarrow R^n - A$$

and

$$j_1 \colon \; sB \longrightarrow B(y_1 ; d_1/2)$$ be the inclusions.

Thus, we have the following diagram.

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "i"'] \arrow[dr, "i_1"] & sB \arrow[dl, "i_2"'] \arrow[d, "i_3"] \arrow[r, "j_1"] & B(y_1 ; d_1/2) \arrow[dl, "j_2"] \\
SB \arrow[r, "i_4"'] & R^n - A &
\end{tikzcd}$$

Then, it is clear that

$$i_3 = j_2 j_1, \quad \text{and}$$

$$i_3^* = j_1^* j_2^*.$$

But

$$i_3 = i_4 i_2 \quad \text{and}$$

$$i_3^* = i_2^* i_4^*.$$

We know that

$$i_1 = i_4 i \quad \text{and}$$

$$i_1^* = i^* i_4^* = s^* i_2^* i_4^* = s^* i_3^* = s^* j_1^* j_2^*.$$

Now j_2^* is trivial, since $B(y_1 ; d_1/2)$ is contractible.

This implies that i_1^* is trivial, a contradiction.

<u>2.2 Definition</u>. Let $(X_w)_{w \text{ in } W}$ be a collection of topological spaces for some index set W, directed by $\leq$.
If $w_1 \leq w_2$, let

$$F_{w_1}^{w_2}: \ X_{w_2} \longrightarrow X_{w_1} \qquad \text{be continuous such that if}$$

$$w_1 \leq w_2 \leq w_3, \text{ then } F_{w_1}^{w_3} = F_{w_1}^{w_2} F_{w_2}^{w_3}, \text{ and}$$

$$F_w^w = \text{identity for all } w \text{ in } W.$$

Let $P_{w \text{ in } W} X_w$ be the Cartesian Product, with the usual product topology.

The subspace X of $P_{w \text{ in } W} X_w$ consisting of all points (x_w) such that if $w_1 \leq w_2$,

$$F_{w_1}^{w_2}(x_{w_2}) = x_{w_1}$$

is the <u>Projective</u> <u>Limit</u> of the system $(X_w, F_{w_1}^{w_2})$ for the directed set W. A compact group is the Projective Limit of Lie groups (6).

$(F_{w_1}^{w_2})$ is the collection of <u>bonding</u> <u>maps</u> of the system.

W is assumed to be directed upwards; that is, given w_1, w_2 in W, there exists w_3 in W such that

$$w_1 \leq w_3 \text{ and } w_2 \leq w_3.$$

There is a canonical map $p_w: \ X \longrightarrow X_w$, which is simply the projection in the w-th coordinate, for each w in W.

$X = A$ is a <u>limit</u> <u>k-manifold</u> if it is compact, connected and each X_w is a compact k-dimensional manifold and

$$p_w^*: \ H^k(X_w; G) \longrightarrow H^k(A; G)$$

is non-trivial for all w in W, where G is torsion-free.

<u>2.3 Theorem</u>. If A is a limit k-manifold such that $cd(A) = k$, each X_w is orientable, and each canonical projection

$$p_w: \ A \longrightarrow X_w$$

is onto, then

$$H^k(A; G) \neq 0 \qquad \text{and}$$

$$H^k(A-U; G) = 0$$

for any non-empty open neighborhood U in A.

<u>Proof</u>. It follows from the definition of a limit k-manifold that

$$H^k(A; G) \neq 0.$$

Let $x = (x_w)$ be a member of A.

Let V' be a member of the usual open basis for the product space

$$P_w \text{ in } {}_W X_w$$

which contains x.

V' is the product of $V_{w_1}, V_{w_2}, \ldots, V_{w_n}$ in the coordinates $w_1, w_2, \ldots, w_n$; and of X_w in all other coordinates.

A typical basic open neighborhood V of x in A is $V' \cap A$.

Let

$$Y_w = \bigcap_{w_i \leq w} (F^w_{w_i})^{-1}(X_{w_i} - V_{w_i}).$$

A-V is the projective limit of $(Y_w)_w$ in W' with the induced bonding maps.

Since the projection maps (p_w) for the system (X_w) are onto, so are the bonding maps for (X_w); thus, if $w_i \leq w$ for some i such that $1 \leq i \leq n$, then

$$(F^w_{w_i})^{-1}(X_{w_i} - V_{w_i})$$

is a proper subset of X_w, and so is Y_w.

Let W' be the set of all elements of W greater than some w_i: $1 \leq i \leq n$. Then, W' is cofinal in W and Y_w is a proper closed subset of X_w for each w in W'.

Now,

$$H^k(Y_w;\ G) = H_o(X_w,\ X_w - Y_w;\ G)$$

for all w in W' (7, Ch. 6, Sec. 9, Thm. 10).

And,

$$H_o(X_w,\ X_w - Y_w;\ G) = 0,$$

since X_w is a connected manifold, hence path-connected, and $X_w - Y_w$ is not empty.

Now

$$H^k(A-V; G) \cong \operatorname*{dir\,lim}_{w \text{ in } W'} H^k(Y_w; G) = 0.$$

If U is any open neighborhood of x, there exists a typical basic open neighborhood $V \subseteq U$, and $H^k(A-V; G) = 0$.

But, $A-U \subseteq A-V$.

Since $cd(A) = k$, $H^k(A-U; G) = 0$.

<u>2.4 Corollary</u>. Let (S, X) be a real normal clan act, such that the H-classes of idempotents in S are connected.

If x is maximal but not minimal in the natural quasi-order, then $x \in$ boundary X.

<u>Proof</u>. If $A = L(x)$, then A is the coset space of a compact, connected group (8). Such a space satisfies the hypotheses of 2.3, (1, 4.1.b), where G is the group of integers. A simple application of the universal coefficient formula yields that the hypotheses of 2.3 are satisfied where G is the field of real numbers. The result then follows from Theorem 2.1.

<u>2.5 Corollary</u>. Let (S, X) be a real normal clan act. If x is maximal but not minimal in the natural quasi-order, then $x \in$ boundary X.

<u>Proof</u>. According to a result of Stadtlander, to appear, which says that given a normal clan act (S, X),

and an irreducible hormos $S' \subsetneq S$, an element which is maximal but not minimal under the action of S, is maximal but not minimal under the action of S', x is maximal but not minimal in the natural quasi-order induced by an irreducible hormos $S' \subsetneq S$. Since S' is an abelian clan with all H-classes of idempotents connected, the result is a consequence of 2.4.

<u>2.6 Corollary</u>. If (S, X) is a real clan act, x is maximal but not minimal in the natural quasi-order, and the component of x in $L(x)$ is open in $L(x)$ and an orientable manifold, then $x \in$ boundary X.

<u>Proof</u>. The proof follows from 2.1 by a slight modification: we call A the component of x in $L(x)$ and choose V_p such that $V_p \cap L(x)$ is in A. We can do this since A is open in $L(x)$. Now, the proof is straightforward, when one takes into account that a compact orientable manifold satisfies the conditions of 1.3.

<u>2.7 Corollary</u>. If (S, X) is a real clan act, x is maximal but not minimal in the natural quasi-order, and the component of x in $L(x)$ is open in $L(x)$ and the coset space of a continuum group, then $x \in$ boundary X.

Proof. As in 2.6, we choose V_p so that $V_p \cap L(x)$ is in A, the component of x in L(x). The fact that A fulfills the conditions of 1.3 is inherent in 2.4.

CHAPTER III

Further Applications

3.1 Remark. A few words here concerning the implications of the above proofs are in order. I should point out that more has been proved above than has appeared in the statements of the theorems and corollaries. Consider, for example, a circle A in R^3.

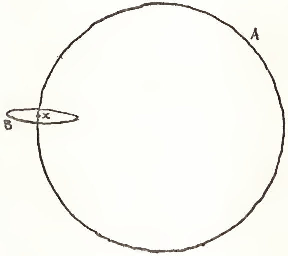

FIGURE II

Then, near each point of A, one can take another circle B which is linked with A like a chain.

Indeed, it requires little imagination to see that one can take a sequence of circles B_i, which converges to x such that each B_i is linked with A. Our way of expressing this is to say that each point of A is <u>locally linked with A in R^n - A</u>. In other words, given any neighborhood V of a point x in A, one can find a circle B in R^3 - A and contained in V such that B is linked with A. The phenomenon of linkage can be characterized by invariants of algebraic topology as follows: B is linked with A iff

$$i: B \longrightarrow R^n - A$$

is the inclusion and there exists some integer j such that

$$i_*: H_j(B) \longrightarrow H_j(R^n - A)$$

is nontrivial.

Similarly, the phenomenon of local linkage can be characterized by these invariants: A is <u>locally linked at x in A</u> iff, given any neighborhood V of x, there exists some integer j such that

$$i_*: H_j(V-A) \longrightarrow H_j(R^n - A)$$

is nontrivial. Now, what 1.2 shows is that this criterion may be replaced by one using A. S. cohomology: namely, that A is locally linked at x in A iff, given any neighborhood V of x, there exists a compact polyhedron B in V-A with

$$i_1: B \longrightarrow R^n - A \quad \text{and for some integer } j$$

$$i_{1*}: \bar{H}^j(R^n - A) \longrightarrow \bar{H}^j(B)$$

is nontrivial.

The advantage in having our linkage criteria expressed in terms of A. S. cohomology derives from the stronger homotopy properties of that theory, which are taken advantage of frequently in subsequent proofs.

Now, 1.3 gives conditions under which a compact set A in R^n is locally linked at a point x in dimension n-k-1. In proofs immediately following this, we show that if A is a compact, connected orientable manifold or the coset space of a continuum group, then it is locally linked at each point.

One needs next to introduce the concept of local linkage of A at x relative to X. A contained in X in R^n is said to be <u>locally linked at x relative to X</u> iff x is in A, and, given any neighborhood of V of x relative to X, there exists a compact polyhedron B in V-A and an integer j such that

$$i_1 : \ B \longrightarrow R^n - A, \qquad \text{and}$$

$$i_1^* : \ \overline{H}^j(R^n - A) \longrightarrow \overline{H}^j(B)$$

is not trivial.

This means one can find the local links with A in X.

Each time we said in our theorems and corollaries above that x is in boundary X, what we actually showed was that A is not locally linked at x relative to X, and that if x were in the interior, it would have to be so linked. In other words, at such points one cannot get a sequence of

links in X approaching x as limit. Thus, we have shown something more about the structure of X at x than simply that x may not be an interior point of X. Let us reformulate the statements of two of the above results in these terms.

3.2 Theorem. Let (S, X) be a real clan act, x be maximal but not minimal in the natural quasi-order; if there exists an integer K, an index set P, and a basis of open n-balls $(V_p)_{p \text{ in } P}$ for the neighborhood system at x such that

$$H^K(A; F) \neq 0 \quad \text{and}$$

$$H^K(A - V_p; F) = 0 \quad \text{for all } p \text{ in } P,$$

then A is not locally linked at x relative to X.

Proof. See 2.1.

3.3 Corollary. Let (S, X) be a real normal clan act, such that the H-classes of idempotents are connected.

If x is maximal but not minimal in the natural quasi-order, then A is not locally linked at x relative to X.

Proof. See 2.4.

3.4 Remark. Now, it should be pointed out that one may have a real normal clan act with A a maximal but not minimal L-class such that A is linked relative to X. Consider the following example, with diagram on the next page, where the

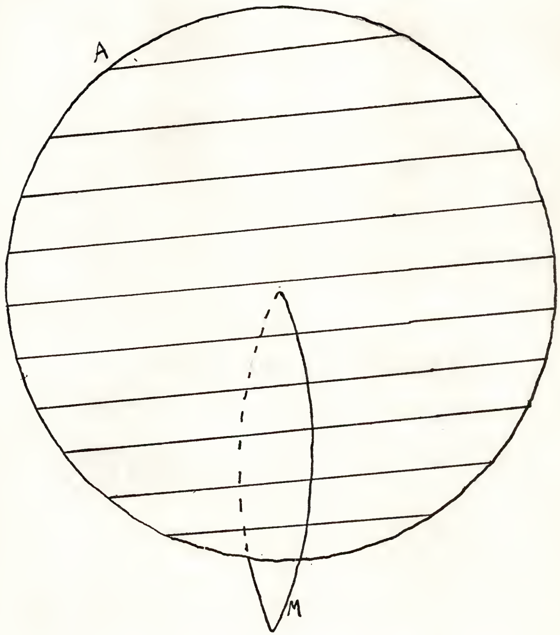

FIGURE III

structure is defined in (3, Chap. D, 2.3.3.3). It is the union of a two-dimensional disc having rim A with a circle M passing through the center of the disc. This is an abelian clan with group of units being A and minimal ideal M, and is, in fact, irreducible with respect to A. We may consider this as both the state-space X and the clan acting on X. It is clear that A is a maximal L-class containing no minimal elements, and that M is linked with A relative to the whole space X. What our theorems tell us about such spaces is that, although such links with A may exist in X, we cannot have a sequence of them converging to a point of A. In other words, the links with A must keep a certain distance away from any point of A.

The fact that, in this example, M is linked with A is not a coincidence. The proof of 2.1 shows that if anything in the state-space is linked with A, then K(S)X, the set of minimal elements in the natural quasi-order, is linked with A.

CHAPTER IV

A Theorem on Quasi-orders

4.1 Remark. Until this stage, we have been concerned with a particular sort of quasi-order on a set X in R^n, namely one induced by the action of a clan on X. One may well ask what conditions may be placed on an arbitrary quasi-order on X which will assure similar results. An answer may be found in 4.2 below. First, we need to expose the terminology to be employed. If Q is a quasi-order on X, then Q is a subset of X x X such that

$$x \; Q \; x \qquad \text{for all } x \text{ in } X$$

and $\quad$ if x Q y and y Q z, then x Q z,

where, as usual, by x Q y is meant (x, y) is in Q. Call H the intersection of Q and Q^{-1}. H is an equivalence relation on X. By H(x) is meant the equivalence class of H containing x. Q and H are said to be real if X is in R^n.

A set S is said to be acyclic in the A. S. cohomology if S has the same cohomology as a point.

The ideal in X generated by A is said to be $[y \mid y$ is in X and y Q a for some a in A$]$.

F is assumed to be a field.

4.2 Theorem. Suppose: Q is a quasi-order on X in R^n, x is maximal in the quasi-order, $H(x) = A$ is compact, there is an integer K, an index set P, and a basis of open n-balls $(V_p)_{p \text{ in } P}$ for the neighborhood system at x such that

$$H^K(A; F) \neq 0 \quad \text{and}$$

$$H^K(A - V_p; F) = 0 \quad \text{for all } p \text{ in } P.$$

If the ideals generated by compact polyhedra in X are acyclic in the (n-K-1)-st dimension, then $x \in$ boundary X.

Proof. Let us suppose, on the contrary, that x is in the interior of X; take V_p such that it contains x and is in the interior of X. Now, we know by 1.3 that there exists a compact polyhedron B in $V_p - A$ such that

$$i_1: \ B \longrightarrow R^n - A \quad \text{and}$$

$$i_1^*: \ \overline{H}^{n-K-1}(R-A; F) \longrightarrow \overline{H}^{n-K-1}(B; F)$$

is not trivial.

Let C be the ideal generated by B. C is, by hypothesis, acyclic in the (n-K-1)-st dimension, so

$$\overline{H}^{n-K-1}(C; F) = 0.$$

Since x is maximal and B is in $R^n - A$, C is in $R^n - A$.

Let $i: \ B \longrightarrow C$ be the inclusion.

Let $i_2: \ C \longrightarrow R^n - A$ be the inclusion.

Clearly $i_1 = i_2 i$.

So $i_1^* = i^* i_2^*$.

$$\begin{tikzcd} \bar{H}^{n-K-1}(R^n - A; F) \arrow[r, "i_2^*"] \arrow[dr, "i_1^*"'] & \bar{H}^{n-K-1}(C; F) \arrow[d, "i^*"] \\ & H^{n-K-1}(B; F). \end{tikzcd}$$

But i_2^* is trivial.

So i_1^* must be trivial, a contradiction.

<u>4.3 Remark</u>. It is possible to extend the above result somewhat by replacing the hypothesis that A be compact with the requirement that A be a closed proper subset of R^n. This is possible since one can alter 1.1, 1.2, and 1.3 by replacing A compact in R^n with A a closed proper subset of R^n, and $H^K(A; F)$, $H^K(A - V_p; F)$ with $H_c^K(A; F)$, $H_c^K(A - V_p; F)$ respectively. The proofs remain identical, after these alterations in the hypotheses.

CHAPTER V

Its Application to Normal Bing Acts

5.1 Remark. We can use 4.2 to obtain results beyond the scope of our previous results on semigroup acts. In what follows, a bing is a compact connected semigroup.

5.2 Theorem. Let (S, X) be a real normal bing act such that the H-classes of idempotents in S are connected, and $K(S)X$ is a point k. If x is maximal in the natural quasi-order, then $x \in$ boundary X.

Proof. We see as in 2.4 that $H(x)$ satisfies the conditions of 4.2. Let B be a compact polyhedron in X. Since S is normal, it is unitary and the ideal generated by B is just SB. Then $K(S)B = k$. Now, by (8), $H^p(SB; F) \cong H^p(K(S)B; F)$ for $p > 0$ and $H^p(K(S)B; F) = H^p(k; F) = 0$ for $p = 0$.

Since $K(S)B$ is a point k and S is connected and unitary, SB is connected; SB is thus acyclic in all dimensions. By 4.2, $x \in$ boundary X.

5.3 Corollary. Let (S, X) be a normal bing act such that X is compact, the H-classes of idempotents

in S are connected and $X/K(S)X$ may be imbedded in R^n. If x is maximal but not minimal in the natural quasi-order, then x does not have a Euclidean n-dimensional neighborhood in X.

Proof. We let $S/K(S)$ act on $X/K(S)X$ in the canonical way. By 5.2, x does not have a Euclidean n-dimensional neighborhood in $X/K(S)X$. Since $K(S)X$ is closed, x does not have a Euclidean n-dimensional neighborhood in X.

5.4 Remark. It is clear that similar results are obtained if, in a real normal bing act (S, X) such that $K(S)X$ is a point k, $L(x)$, for x maximal but not minimal in the natural quasi-order, is a connected orientable manifold or the coset space of a continuum group; or if the component of x in $L(x)$ is of this sort and is open in $L(x)$.

CHAPTER VI

Spherical Acts

6.1 Remark. It may sometimes be of interest to work in S^n rather than in R^n, because of the compactness of S^n. If one reviews the results of the previous chapters in this light, one finds that all of them can be established for S^n, if A is a closed proper subset of S^n. Theorem 1.3 is the only one requiring a substantially different proof from the real case.

Something to bear in mind in applying these results about spherical acts is that the case of interest is when X is a proper subset of S^n, and, hence, in R^n anyway. For, if S is a clan or a normal bing and $X = S^n$, then a cohomology argument shows that $K(S)X = X$, and all elements are minimal.

CHAPTER VII

An Intersection Theorem in Clan Acts

In (9), A. D. Wallace proves the following.

7.1 Theorem. If S is a connected compact topological semigroup with left unit, A a closed subset of S, $T = (x \mid xA \cap A$ is not empty$)$ is a proper subset of S, and $cd(S, G) \leq n$, then $H^n(A; G) = 0$.

Proof. Take U open such that T is in U and $\overline{U}$, the closure of U, is properly contained in S. Let q be a left unit. Then q is in T.

Let C be the component of U including q.

Then $z = \overline{C}$ is a compact connected subset of S meeting $\overline{U}-U$ in a point p in $S-T$.

Thus, $A \cap pA$ is empty.

Let $a\colon A \longrightarrow A \cup pA$.

$b\colon pA \longrightarrow A \cup pA$.

$C\colon A \cup pA \longrightarrow ZA$

be the inclusion maps.

$a_0\colon A \longrightarrow A$ is inclusion or multiplication by q.

$b_0\colon A \longrightarrow pA$ is multiplication by p.

m_o: $A \rightarrow ZA$ is inclusion, or multiplication by q.

m_1: $A \rightarrow ZA$ is multiplication by p.

We have the following diagram:

$$\begin{tikzcd}
 & & H^n(A) \arrow[dll, "a_o^*"] \\
H^n(A) & H^n(ZA) \arrow[l, "m_o^*"', "m_1^*"] \arrow[r, "c^*"] & H^n(A \quad pA) \arrow[u, "a^*"'] \arrow[d, "b^*"] \\
 & & H^n(pA) \arrow[ull, "b_o^*"']
\end{tikzcd}$$

Brief examination shows that $m_o^* = m_1^*$, due to the Homotopy Lemma, and the diagram commutes.

From the Excision Theorem, we know that

$$a^* + b^*: \ H^n(A \cup pA) \rightarrow H^n(A) \oplus H^n(pA)$$

is an isomorphism.

If h is in $H^n(A)$, let h_1 be in $H^n(A \cup pA)$ such that

$$a^*(h_1) = h \text{ and } b^*(h_1) = 0.$$

Since Z and A are compact and $cd(S, G) \leq n$, there exists h_2 in $H^n(ZA)$ such that

$$c^*(h_2) = h_1.$$

Now, $m_o^*(h_2) = h$, and $m_1^*(h_2) = 0$.

Since $m_o^* = m_1^*$, $h = 0$.

<u>7.2 Remark</u>. Theorem 7.1 has a natural application to clan acts if the state-space has codimension n, and nontrivial cohomology in the n-th dimension for some abelian coefficient group G. We can lead up to this as follows.

<u>7.3 Theorem</u>. Let (S, X) be a clan act, where X is compact, T_2 and for some abelian coefficient group G, $cd(X; G) \leq n$. If A is a closed subset of X and $T = (s | s$ is in S and sA does intersect A$)$ is a proper subset of S, then $H^n(A; G) = 0$.

<u>Proof</u>. The proof is done just as in 7.1.

<u>7.4 Corollary</u>. Suppose (S, X) is a clan act, $cd(X; G) = n$, and $H^n(X; G) \neq 0$.

Then, for all s, t in S, sX intersects tX.

<u>Proof</u>. Since $H^n(X; G) \neq 0$ and S is a clan, our old friend, the Homotopy Lemma, gives us that $H^n(tX; G) \neq 0$.

Now, we apply 7.3, letting tX take the place of A. We see that T must be the entire semigroup S. Thus, stX intersects tX.

Since stX is in Sx, sX intersects tX.

CHAPTER VIII

Another Intersection Theorem

The result of 7.4 is not surprising if S is normal, for we have the following.

8.1 Theorem. If (S, X) is a normal compact act for X compact T_2, then

$$\bigcap_{s \text{ in } S} sX = pX$$

where p is any member of K(S), the minimal ideal of S.

Proof. Suppose p_1, p_2 are in K(S). Since S is normal, K(S) is a group.

$$p_1X = p_2p_2^{-1}p_1X \text{ is in } p_2X.$$

Similarly,

$$p_2X \text{ is in } p_1X.$$

Thus, $\quad p_1X = p_2X$ for any p_1, p_2 in K(S).

Further, if p is in K(S), then sp is in K(S) for any s in S.

Thus, pX = spX is in sX.

8.2 Corollary. If (S, X) is a normal compact act for X compact T_2, then sX intersects tX for any s, t in S.

Proof. Follows directly from 8.1.

8.3 Theorem. If (S, X) is a compact act, for X compact T_2, and if E, the set of idempotents in S, is commutative, then

$$\bigcap_{s \text{ in } S} sX = \bigcap_{e \text{ in } E} eX \quad \text{is not empty.}$$

Proof. First, as to showing $\bigcap_{e \text{ in } E} eX$ is not empty. Take $e_1, e_2, \ldots, e_n$ in E.

Then, $e_1X \cap e_2X \cap \ldots \cap e_nX$ contains $e_1e_2 \ldots e_nX$ by a simple argument relying on commutativity.

The finite intersection property then yields that

$$\bigcap_{e \text{ in } E} eX \text{ is not empty.}$$

Clearly, $\bigcap_{s \text{ in } S} sX$ is in $\bigcap_{e \text{ in } E} eX$.

In order to prove the reverse inequality, it is enough to show that for any s in S, there exists e in E such that sX contains eX.

One merely takes e to be the unique idempotent of the minimal ideal in the closure of the semigroup generated by s (11).

CHAPTER IX

Convergence of L-Classes

9.1 Remark. In this chapter, we shall let (S, X) be a normal compact act, for X compact T_2. Stadtlander (8) shows that, in this case, there exists a function

$$\varphi : X/L \rightarrow E(S)$$

with the property that, if x is in X and $\bar{x}$ in X/L is p(x) for $p: X \rightarrow X/L$ the canonical projection, then $(\bar{x})$ is the unique least idempotent e in S such that ex = x.

Let $F = \varphi \circ p: X \longrightarrow E(S)$.

F need not, in general, be continuous. An example suggested by Stadtlander is the unit interval acting on itself under the usual multiplication. A diagram appears on the next page.

We let $x_i = 1/i$, i = 1, 2, ... Clearly, (x_i) converges to 0. There are only two idempotents, 0 and 1; $F(x_i) = 1$ for all i. Thus, $(F(x_i))$ converges to 1. But, $F(0) = 0$.

However, the function F is, in fact, lower semi-continuous. By this is meant simply that for any idempotent e in E(S), $(x\ F(x) \leq e)$ is closed.

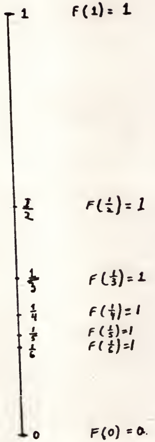

FIGURE IV

Recall that, in $E(S)$, $e_1 \leq e_2$ iff $e_1e_2 = e_1$, and since S is normal, $E(S)$ is commutative.

<u>9.2 Theorem</u>. Let (S, X) be a normal compact act for X compact T_2. $F: X \longrightarrow E(S)$ is lower semicontinuous.

<u>Proof</u>. Suppose $(x_a)_{a \text{ in } A}$ is a net in X converging to x in X, such that, for some e in $E(S)$,

$$F(x_a) \leq e \qquad \text{for all } a \text{ in } A.$$

Then, $ex_a = e(F(x_a)x_a) = (eF(x_a))x_a$

$$= F(x_a)x_a = x_a \qquad \text{for all } a \text{ in } A.$$

By the continuity of the act, ex_a converges to ex.

Since $ex_a = x_a$, $ex = x$.

Thus, $F(x) \leq e$.

<u>9.3 Remark</u>. Given a net $(x_a)_{a \text{ in } A}$ converging to x, it is easily seen that $(L(x_a))_{a \text{ in } A}$ need not converge to $L(x)$. Consider the example of a cone with a wind having an initial point acting on a sequence of such objects converging to a base cone. A diagram appears on the next page.

For all i, $L(x_i) = x_i$.

However, $L(x)$ is the circle forming the upper rim of the base cone.

It is clear that $\text{Lim } L(x_i) = x$, which is properly contained in $L(x)$.

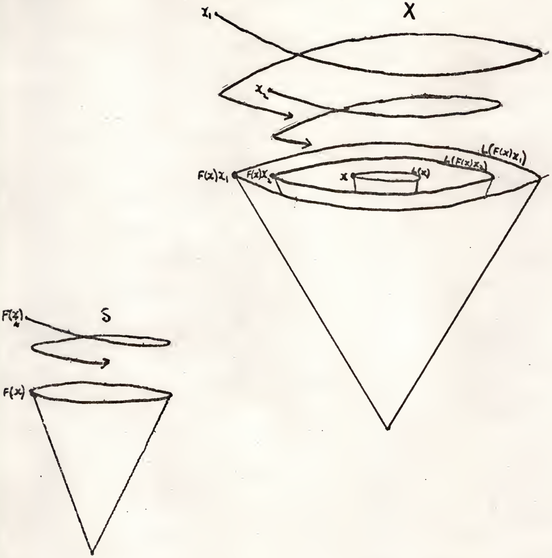

FIGURE V

One notices, though, that $(F(x)x_i)$ converges to x and that $\operatorname{Lim}_i L(F(x)x_i) = L(x)$. Stadtlander conjectured that this is true in general. Such is, in fact, the case, as we see in the following.

<u>9.4 Theorem</u>. Let (S, X) be a normal compact act, for X compact T_2. Let $(x_a)_{a \text{ in } A}$ be a net in X converging to x in X. Then

$$\operatorname{Lim}_{a \text{ in } A} L(F(x)x_a) = L(x).$$

<u>Proof</u>. We shall prove that

a) $L(x)$ is contained in $\operatorname{Lim\,inf}_{a \text{ in } A} L(F(x)x_a)$

and b) $\operatorname{Lim\,sup}_{a \text{ in } A} L(F(x)x_a)$ is contained in $L(x)$.

Since Lim inf is contained in Lim sup, this will show that $\operatorname{Lim}_{a \text{ in } A} L(F(x)x_a)$ exists and equals $L(x)$.

a) y is in $\operatorname{Lim\,inf}_{a \text{ in } A} L(F(x)x_a)$ iff, given any neighborhood V of y, there exists a_o in A such that, for all $a \geq a_o$, V intersects $L(F(x)x_a)$.

Take any y in $L(x)$ and any neighborhood V in X containing y.

We know from (8) that $H(F(x))$ in S acts transitively on $L(x)$. Thus, there exists h_1 in $H(F(x))$ such that

$$h_1x = y.$$

By the continuity of the act, we select a neighborhood V_1 of x such that h_1V_1 is contained in V.

Now, $(F(x)x_a)_{a \text{ in } A}$ converges to $F(x)x = x$.

Thus, there exists a_o in A such that, for all $a \geq a_o$,

$F(x)x_a$ is in V_1.

So, $h_1F(x)x_a$ is in V for all $a \geq a_o$.

Since $H(F(x))$ is a group, it is straightforward that $H(F(x))\, F(x)x_a$ is in $L(F(x)x_a)$; Part a) has thus been proven.

b) y is in $\operatorname{Lim\,sup}_{a \text{ in } A} L(F(x)x_a)$ iff, given any neighborhood V of y and any a_o in A, there exists $a_1 \geq a_o$ such that $L(F(x)x_{a_1})$ intersects V.

Take any y in $\operatorname{Lim\,sup}_{a \text{ in } A} L(F(x)x_a)$.

We wish to show that there exist elements s, t in S such that $sy = x$ and $y = tx$, which together imply that y is in $L(x)$.

By hypothesis, there exists a net $(y_a)_{a \text{ in } A}$ such that, for all a, y_a is in $L(F(x)x_a)$ and y is a cluster point of $(y_a)_{a \text{ in } A}$.

Thus, there exists a subnet $(y_b)_{b \text{ in } B}$ of $(y_a)_{a \text{ in } A}$ such that $(y_b)_{b \text{ in } B}$ converges to y, where B is chosen merely as an appropriate cofinal subset of A.

Since y_b is in $L(F(x)x_b)$, there exist s_b and t_b in S such that $s_b y = F(x)x_b$ and $y = t_b F(x)x_b$.

Since S is compact, there exist convergent subnets $(s_c)_{c \text{ in } C}$ and $(t_d)_{d \text{ in } D}$ of $(s_b)_{b \text{ in } B}$ and $(t_b)_{b \text{ in } B}$, respectively, where C and D are cofinal subsets of B and hence of A.

Call s the limit of $(s_c)_{c \text{ in } C}$ and t the limit of $(t_d)_{d \text{ in } D}$.

Clearly, $(x_c)_{c \text{ in } C}$ and $(x_d)_{d \text{ in } D}$ converge to x.

Now, the continuity of the act allows us to take limits on both sides of the equations:

$$s_c y = F(x)x_c \qquad \text{for all c in C.}$$

$$y = t_d F(x)x_d \qquad \text{for all d in D.}$$

We get

$$sy = x$$

$$y = tx.$$

CHAPTER X

More Convergence Theorems

<u>10.1 Remark</u>. In this chapter, we shall be dealing only with abelian compact acts. We find that, under some conditions, when a net $(x_a)_{a \text{ in } A}$ converges to x, the L-classes $L(x_a)_{a \text{ in } A}$ do indeed converge to L(x). The conditions are essentially that the net approaches x from below. Indeed, L(x) turns out to be the projective limit of $L(x_a)_{a \text{ in } A}$ with suitable bonding maps. This result was conjectured by Stadtlander, who encouraged me to prove it; he had a different proof of 10.3.

M(x) is defined to be $(s|s$ is in S, and sL(x) is in L(x)). This is shown in (8) to be $(s|s$ is in S, and sL(x) intersects L(x)) and $(s|s$ is in S, and $sL(x) = L(x))$.

M(x) is a compact subsemigroup of S, and H(F(x)) is the minimal ideal of M(x). H(F(x)) acts transitively on L(x).

It is easy to show that, if $x \leq y$, then M(x) contains M(y), and $F(x) \leq F(y)$.

Let S_x denote the stability subgroup of x in H(F(x)).

Suppose now that $x_a \leq x_b$. Then

$$m_a^b\colon\ H(F(x_b)) \longrightarrow H(F(x_a)) \qquad \text{is defined by}$$

$$m_a^b(h) = h \cdot F(x_a)$$

since $H(F(x_b))$ is in $M(x_b)$, which is contained in $M(x_a)$, and $M(x_a) \cdot F(x_a)$ is in $H(F(x_a))$.

Furthermore, this is a continuous homomorphism. Since $S_{x_b} \cdot F(x_a)$ is in S_{x_a}, a function

$$\overline{m}_a^b\colon\ H(F(x_b))/S_{x_b} \longrightarrow H(F(x_a))/S_{x_a}$$

is well defined and continuous.

Let $r_a\colon\ H(F(x_a))/S_{x_a} \longrightarrow L(x_a)$ be the canonical homeomorphism.

Define $n_a^b\colon\ L(x_b) \longrightarrow L(x_a)$ as $r_a \overline{m}_a^b r_b^{-1}$.

We have the following commutative diagram, given $x_a \leq x_b$.

$$\begin{tikzcd}
H(F(x_b)) \arrow[r, "p_b"] \arrow[d, "m_a^b"] & H(F(x_b))/S_{x_b} \arrow[r, "r_b"] \arrow[d, "\overline{m}_a^b"] & L(x_b) \arrow[d, "n_a^b"] \\
H(F(x_a)) \arrow[r, "p_a"'] & H(F(x_a))/S_{x_a} \arrow[r, "r_a"'] & L(x_a)
\end{tikzcd}$$

<u>10.2 Theorem</u>. Let (S, X) be a compact abelian act, for X compact T_2. Given a net $(x_a)_{a \text{ in } A}$ converging to x such that $a \leq b$ implies $x_a \leq x_b$, then L(x) is the projective limit of the system $(L(x_a))_{a \text{ in } A}$ with bonding maps n_a^b.

<u>Proof</u>. For any a in A, let $A_a = (a' | a'$ is in A and $a' \geq a)$. Clearly, A_a is cofinal in A. $x_{a'} \geq x_a$ for all a' in A_a. So $x_a = s_{a'}x_{a'}$ for all a' in A_a.

Since S is compact, consider a subnet of $s_{a'}$ converging to s_a in S. Then, by continuity of the act, $x_a = s_a x$. Thus, $x_a \leq x$ for all a in A.

$$\text{Define} \quad p: \; H(F(x)) \longrightarrow H(F(x))/S_x$$

$$r: \; H(F(x))/S_x \longrightarrow L(x)$$

$$n_a: \; L(x) \longrightarrow L(x_a) \text{ by } n_a = r_a \overline{m}_a r^{-1}$$

where

$$m_a: \; H(F(x)) \longrightarrow H(F(x_a)) \quad \text{and}$$

$$\overline{m}_a: \; H(F(x))/S_x \longrightarrow H(F(x_a))S_{x_a}$$

are all defined as above.

Now, define $n: \; L(x) \longrightarrow P_{a \text{ in } A} L(x_a)$ by requiring the projection in the a-th coordinate to be n_a.

Call D the projective limit of the system $L(x_a)$ with bonding maps n_a^b.

If $a \leq b$, then $\overline{m}_a^b \overline{m}_b = \overline{m}_a$, since $F(x_b) \cdot F(x_a) = F(x_a)$.

Thus, for $a \leq b$,

$$n_a^b n_b = r_a \overline{m}_a^b r_b^{-1} r_b \overline{m}_b r^{-1}$$

$$= r_a \overline{m}_a^b \overline{m}_b r^{-1}$$

$$= r_a \overline{m}_a r^{-1} = n_a.$$

Thus, the image of n is contained in D.

n: $L(x) \longrightarrow D$ is a continuous homomorphism, since its projections are continuous homomorphisms.

It remains to show that a) n is one-to-one, and b) n is onto D.

a) Suppose y_1, y_2 are in $L(x)$, and $n(y_1) = n(y_2)$. Let s_1 be in $p^{-1}r^{-1}(y_1)$, s_2 be in $p^{-1}r^{-1}(y_2)$.

$$n_a(y_1) = s_1 \cdot F(x_a)x_a = s_2 F(x_a)x_a = n_a(y_2).$$

Hence, $\quad s_1 x_a = s_2 x_a \quad$ for all a in A.

Thus, $s_1 x = s_2 x$.

But, s_1, s_2 are in $H(F(x))$, and so

$s_1 + S_x = s_2 + S_x$, where $s_i + S_x$ is the coset of S_i in $H(F(x))$ modulo S_x, $i = 1, 2$.

This implies $r^{-1}(y_1) = r^{-1}(y_2)$.

Since r is one-to-one, $y_1 = y_2$.

b. Let (y_a) be in D. Let s_a be in $p_a^{-1} r_a^{-1}(y_a)$. Then, $s_a x_a = y_a$ for all a in A. (y_a) must converge. For

suppose not, Then, there exist two distinct cluster points w, z. There exists a subnet $(s_b x_b)_{b \text{ in } B}$ converging to w, and a subnet $(s_c x_c)_{c \text{ in } C}$ converging to z, where B and C are cofinal in A such that (s_b) converges to s_1 and (s_c) converges to s_2.

Then, $w = s_1 x$

$z = s_2 x.$

Define $N: B \longrightarrow C$ such that $N(b) \geq b$. Then, $N(B)$ is cofinal in C. And, $(s_{N(b)} x_b)_{b \text{ in } B}$ converges to $s_2 x = z$.

Now, $s_{N(b)} x_b = s_{N(b)} F(x_b) x_b$. And $s_{N(b)} F(x_b) x_b = n_b^{N(b)}(s_{N(b)} x_{N(b)}) = s_b x_b$, since $(s_a x_a)$ is in D.

Thus, $(s_{N(b)} x_b) = (s_b x_b)$, which converges to w.

And so $w = z$.

So, we have seen that $(y_a)_{a \text{ in } A}$ must converge to some element, which we call y.

It is not difficult to show that y is in $L(x)$. We have already seen that $y = sx$ for some s in S. Similarly, $t_a y_a = x_a$ for all a in A. Take a subnet of (t_a) which converges to t. Then, $ty = x$.

Now, we claim that $n(y) = (y_a)_{a \text{ in } A}$, or, in other words, that $n_a(y) = y_a$.

Let s_a be in $p_a^{-1} r_a^{-1}(y_a)$.

Then, $s_a x_a = y_a$.

Choose a cofinal subset B such that $(s_b)_{b \text{ in } B}$ converges to some element in S; call it s. Then, y = sx. So, s is in M(x). Thus, sF(x) is in H(F(x)). Hence, sF(x) is in $p^{-1}r^{-1}(y)$. As above, we see that

$$s_b x_{b_o} = s_{b_o} x_{b_o} \qquad \text{for all } b \geq b_o.$$

Thus, $sx_{b_o} = s_{b_o} x_{b_o}$ and $sF(x)x_{b_o} = s_{b_o} x_{b_o}$.

$$n_{b_o}(y) = sF(x)F(x_{b_o})x_{b_o}$$

$$= sx_{b_o} = s_{b_o} x_{b_o} = y_{b_o},$$

where b_o is any fixed member of B.

Since B is cofinal in A, we can conclude that $n_a(y) = y_a$ for all a in A.

10.3 Corollary. Let (S, X) be an abelian compact act, for X compact T_2.

If, in a net $(x_a)_{a \text{ in } A}$ converging to x, $a \leq b$ implies $x_a \leq x_b$, then

$$L(x) = \lim_{a \text{ in } A} L(x_a).$$

Proof. From the proof of 10.2, we know that L(x) is a subset of $\liminf_{a \text{ in } A} L(X_a)$.

In fact, $n(y)$ is a net converging to y for y in $L(x)$.

It is a simple matter to show that

$$\operatorname*{Lim\,sup}_{a \text{ in } A} L(x_a)$$

is in $L(x)$.

CHAPTER XI

Further Theorems on L-Classes

11.1 Remark. We now consider two results about L-classes, one relating the codimension of L-classes to the codimension of a compact abelian semigroup S acting on X, one concerning the topology of L(x). It should be noted that the same methods are used to obtain a result similar to 11.2 in (8).

11.2 Theorem. If (S, X) is an abelian compact act, then

$$\mathrm{cd}(S) \geq \max\ (\mathrm{cd}(L(x)) \mid x \text{ is in } X).$$

Proof. H(F(x)) acts transitively on L(x); thus, L(x) is a homomorphic image of H(F(x)). And so cd(H(F(x))) $\geq$ cd(L(x)). But cd(S) $\geq$ cd(H(F(x))).

11.3 Theorem. If (S, X) is a compact act, X is connected, and L(x) is properly contained in X for x maximal in the natural quasi-order, then L(x) is nowhere dense in X.

Proof. Suppose, on the contrary, that y is in V in L(x), and V is open in X.

Let z be any element of L(x).

From the definition of an L-class, there exists s in S such that sz = y.

By the continuity of the act, there exists W open in X such that s $\cdot$ W is in V.

Since x is maximal in the quasi-order, W is in L(x).

Thus, L(x) is open in X; since L(x) is compact, it is closed in X. This contradicts the connectedness of X.

CHAPTER XII

Applications of a Theorem of Jane Day

In a recent paper (1), Jane Day proves the following.

12.1 Theorem. Let S be a clan acting on a continuum X such that $cd(X) = n$. There is a minimal S-ideal A in X with $H^n(A) \neq 0$ iff $H^n(Gx) \neq 0$ for some x in X and maximal group G in K(S). If there is such a minimal S-ideal, then A is the only minimal S-ideal and is also the unique floor for every nonzero element of $H^n(X)$.

Here, of course, A being an S-ideal means just that SA is in A. By A being a floor for a nonzero element h in $H^n(X)$ is meant that A is closed, $h|A \neq 0$, and for any closed proper subset B in A, $h|B = 0$.

Theorem 12.1 follows easily from the next two theorems.

12.2 Theorem. Let S be a clan acting on a continuum X with $cd(X) \leq n$. If $H^n(A) \neq 0$ for some closed subset A of X, then A contains an S-ideal. If A

is a floor for any nonzero element of $H^n(X)$, then $sA = A$ for each s in S.

<u>12.3 Theorem</u>. Suppose (G, X) is a transformation group action, where G and X are continua. If $cd(X) = n$ and $H^n(Gx) \neq 0$ for some x in X, then the act is transitive.

<u>12.4 Remark</u>. Theorem 12.2 is due to A. D. Wallace in (11); Theorem 12.3 to Jane Day in (1). The proof of the latter is quite long. These results have some further consequences for acts, which we shall discuss.

In what follows, the orbit of x is simply Sx.

<u>12.5 Theorem</u>. Suppose (S, X) is a normal clan act for X compact T^2, $cd(X) \leq n$, and the orbit of some element of X has nontrivial n-th cohomology group. Then, the following are true:

a.) $cd(X) = n = cd(K(S)X)$

b.) $H^n(X) \cong H^n(K(S)X) \neq 0$.

<u>Proof</u>. Let $i: Sx \rightarrow X$ be the inclusion, and $H^n(Sx) \neq 0$. Since X has $cd \leq n$, $i^*: H^n(X) \longrightarrow H^n(Sx)$ is onto.

Thus, $H^n(X) \neq 0$, and $cd(X) = n$.

It is well known that $H^*(X) \cong H^*(K(S)X)$.

So, $H^n(K(S)X) \neq 0$.

Since $K(S)X$ is contained in X, $cd(K(S)X) \leq n$, and is, in fact, n, by virtue of the preceding line.

12.6 Theorem. If (S, X) is a normal clan act, X is a continuum with $cd(X) \leq n$, and the orbit of some element x in X has nontrivial n-th cohomology group, then

a) The continuum topological group $K(S)$ acts transitively on the continuum $K(S)X$.

b) The cohomology groups of the orbits of any two elements are isomorphic.

c) $H^n(Sy) \neq 0$ for all y in X.

d) $H^n(X) \cong H^n(K(S)X) \neq 0$, $cd(X) = n = cd(K(S)X)$.

e) $K(S)X$ is the unique minimal ideal of X, and is the floor for all nonzero elements of $H^n(X)$.

f) If A is a closed proper subset of $K(S)X$, then $H^n(A) = 0$.

Proof. a) Since $H^*(Sx) \cong H^*(K(S)x)$, $H^n(K(S)x) \neq 0$. Consider now the transformation group action $(K(S), K(S)X)$. Let e be the identity of $K(S)$. Then, ex is in $K(S)X$, and $K(S)ex = K(S)x$. So, $H^n(K(S)ex) \neq 0$. We may now apply 12.3.

b) Let x, y be in X. We know that $H^*(Sx) \cong H^*(K(S)x)$ and $H^*(Sy) \cong H^*(K(S)y)$, where these isomorphisms are induced by inclusion. Now, $K(S)x = K(S)ex = K(S)X = K(S)ey = K(S)y$. So, $H^*(Sx) \cong H^*(Sy)$.

c) This follows from b, and the hypothesis that $H^n(Sx) \neq 0$.

d) This follows from 12.5.

e) The maximal group G in K(S) is, of course, K(S), since S is normal. Since the hypotheses of 12.1 are satisfied, we may apply it to obtain our result.

f) This follows from e, the definition of a floor, and the fact that $cd(X) = n$.

We say that a semigroup act is <u>effective</u> provided $sx = tx$, for all x in X implies $x = t$, where s and t are in S.

<u>12.7 Theorem</u>. If (S, X) is an effective abelian clan act, X is a continuum, $cd(X) \leq n$, and, for some x in X, $H^n(Sx) \neq 0$, then K(S) is homeomorphic to K(S)X.

<u>Proof</u>. First, we show that K(S) is effective on K(S)X. Suppose k_1, k_2 are in K(S) and $k_1x = k_2x$ for all x in K(S)X. Let e be the identity of K(S).

Then, for all y in S,

$$k_1y = (k_1e)y = k_1(ey) = k_2(ey) = (k_2e)y = k_2y.$$

Since S is effective on X, $k_1 = k_2$.

Define $f: K(S) \to K(S)X$ by $f(k) = kx$. We know by 12.6 that f is onto. If

$$k_1x = k_2x, \text{ then } k_1kx = kk_1x = kk_2x = k_2kx.$$

Since any element of K(S) can be represented by kx, and K(S) is effective on K(S)X, then $k_1 = k_2$.

<u>12.8 Theorem</u>. If (S, X) is an abelian bing act for X compact T^2, and $H^n(S_x) \neq 0$ for some x in X, then $cd(S) \geq n$.

<u>Proof</u>. $H^n(Sx) \cong H^n(K(S)x)$ by (8). Thus, $H^n(K(S)x) \neq 0$, and so $cd(K(S)x) \geq n$.

K(S)x is the homomorphic image of K(S); so $cd(K(S) \geq cd(K(S)x) \geq n$. Thus, $cd(S) \geq cd(K(S)) \geq n$.

CHAPTER XIII

Algebraic Acts

The following theorem of Hosszu (4) has application to strictly algebraic acts.

13.1 Theorem. Every transitive system of commutable mappings $(F_a | a \text{ is in } A)$ of a set X has the form $x \rightarrow F_a(x) = x + F(a)$ for some a in A where + is an abelian group operation on X and $F: A \rightarrow X$ is onto.

A system of mappings $x \rightarrow F_a(x)$ is transitive if, for all x in X, $F_A(x) = X$, where

$$F_A(x) = (F_a(x) | a \text{ is in } A).$$

It is commutable if $F_a F_b(x) = F_b F_a(x)$ for all a, b in A and x in X.

In the proof of this theorem, $F: A \rightarrow X$ is defined by choosing a fixed x_o in X and letting $F(a) = F_a(x_o)$. Thus, our choice of F is, to a certain extent, arbitrary. The group operation + is defined on X as follows: given x, y in X, there exists a in A such that $y = F(a)$.

Then, $x + y = F_a(x)$.

It is proven that, given a choice of F, the definition of $x + y$ is independent of the choice of a in $F^{-1}(y)$, and that + is, in fact, an abelian group operation.

We say that a semigroup S acts transitively on X if, for any x in X, $Sx = X$.

13.2 Theorem. If S is an abelian semigroup which acts transitively on X, then X is an abelian group and is the homomorphic image of S.

Proof. Here, S corresponds to A in 13.1, and F_s: $X \to X$ is given by $x \to s \cdot x$. Clearly, the hypotheses of 13.1 are satisfied, and X is an abelian group under +, once we have fixed a choice of F: $S \to X$.

Let us fix such a function, F, by selecting x_o in X. Thus, $F(s) = sx_o$.

Suppose $F(s) = x$ and $F(t) = y$ for s, t in S. Then, $x + y = x + F(t) = F_t(x) = tx = t(sx_o) = (ts)x_o = (st)x_o$ $= F(st)$.

13.3 Theorem. If S is an abelian semigroup which acts transitively and effectively on X, then S and X are isomorphic abelian groups.

Proof. It remains only to show that F in 13.2 is one-to-one.

Suppose $F(s) = F(t)$ for s, t in S. Then, $sx_0 = tx_0$. For any element r in S, $r(sx_0) = r(tx_0)$. Thus, $s(rx_0) = t(rx_0)$.

Since S is transitive, $sx = tx$ for all x in X. And so, effectiveness yields that $s = t$.

CHAPTER XIV

A Wedge Theorem for Acts

The following result for semigroups is known as the Wedge Theorem (3, Chap. B, Sec. 6, Them. 16).

14.1 Theorem. Let S be a compact semigroup with identity such that the component of the identity meets the minimal ideal. Let H be any H-class of S which is outside K(S), and let h belong to H. Then, there exists a compact connected subspace T which has the following properties:

a) $T \cap H = h$

b) $T \cap K(S)$ is not empty

c) T is in Sh.

Based on this, one may prove in a direct way the following result for normal bing acts.

14.2 Theorem. Let (S, X) be a normal bing act, and x belong to X - K(S)X. Then, there exists a continuum T in X such that

a) $T \cap L(x) = x$

b) $T \cap K(S)X$ is not empty

c) T is in F(x)X, where $F: X \to E(S)$ is described above.

One may use 14.2 in classifying the possible normal bing acts on continua in R^2. It is certainly helpful in attempting such a classification in R^3, although this project is not complete.

A list of the former appears on the next page; one may, in general, adjoin stickers (closed intervals homeomorphic to the unit interval) to them.

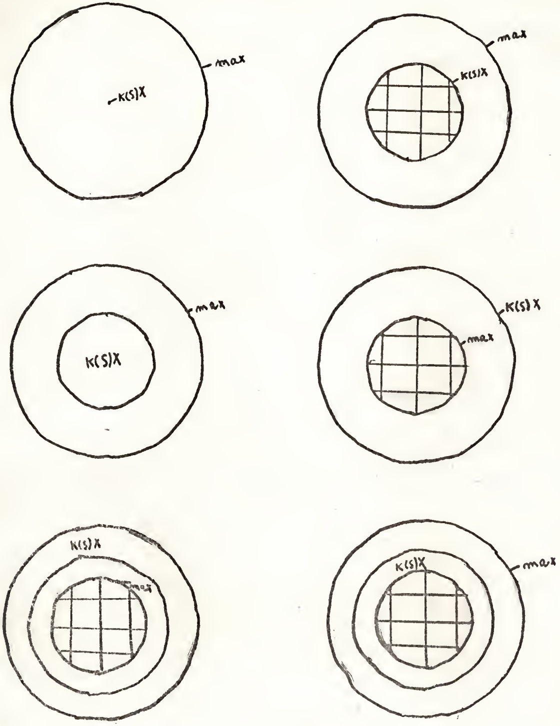

FIGURE VI

BIBLIOGRAPHY

(1) J. Day, "Clan Acts and Codimension," to appear.

(2) M. J. Greenberg, Lectures on Algebraic Topology, W. A. Benjamin, New York, 1967.

(3) K. H. Hofmann and P. S. Mostert, Elements of Compact Semigroups, Charles E. Merrill Bks, Columbus, Ohio, 1966.

(4) M. Hosszu, "Notes on Commutable Mappings," Publ. Math., Tomus 9, Fasc. 1-2 (1962), pp. 105-106.

(5) W. S. Massey, Homology and Cohomology Theory, Yale Lecture Notes, New Haven, 1964.

(6) D. Montgomery and L. Zippin, Topological Transformation Groups, Interscience, New York, 1966.

(7) E. H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1965.

(8) D. Stadtlander, "Semigroup Actions and Dimension," Aeq. Math., Vol. 3, fasc. 1/2 (1969), pp. 1-14.

(9) A. D. Wallace, "Cohomology, Dimension, and Mobs," Summa Brasil. Math, Vol. 3, fasc. 5 (1953), pp. 42-55.

(10) A. D. Wallace, "Inverses in Euclidean Mobs," Math. Journ. of Okayama Univ., Vol. 3 (1953), pp. 23-28.

(11) A. D. Wallace, Project Mob, Univ. of Fla. Math. Dept., Gainesville, 1965.

BIOGRAPHICAL SKETCH

Prior to enrolling in the Graduate School at the University of Florida in 1969, Clark McGranery received an A.B. in Mathematics at the University of Notre Dame, and served in the U. S. Army. He is twenty-nine.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

A D Wallace

A. D. Wallace, Chairman
Graduate Research Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David Stadtlander

Dave Stadtlander
Assistant Professor, Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

E C Nummela

E. C. Nummela
Assistant Professor, Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Thomas Bowman

Thomas Bowman
Assistant Professor, Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Kermit N. Sigmon

Kermit N. Sigmon
Assistant Professor, Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Larry Hench
Larry Hench
Associate Professor, Metallurgical and Materials Engineering

This dissertation was submitted to the Department of Mathematics in the College of Arts and Sciences, and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June, 1972

Dean, Graduate School